

The local asymptotic estimation for the supremum of a random walk with generalized strong subexponential summands *

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Abstract

In this paper, the local asymptotic estimation for the supremum of a random walk and its applications are presented. The summands of the random walk have common long-tailed and generalized strong subexponential distribution. This distribution class and the corresponding generalized local subexponential distribution class are two new distribution classes with some good properties. Further, some long-tailed distributions with intuitive and concrete forms are found, which show that the intersection of the two above-mentioned distribution classes with long-tailed distribution class properly contain the strong subexponential distribution class and the locally subexponential distribution class, respectively.

Keywords: random walk; supremum; local asymptotic estimation; generalized strong subexponential distribution; generalized locally subexponential distribution

2000 Mathematics Subject Classification: Primary 60E05; Secondary 60F99

1 Introduction

In this paper, we primarily study the local asymptotics for the supremum of a random walk generated by summands with common long-tailed and generalized strong subexponential distribution. The generalized strong subexponential distribution class and corresponding generalized local subexponential distribution class are two new distribution classes with some good properties. Therefore, we introduce some related concepts and notations in this section.

Unless otherwise stated, we always assume that a random variable (r.v.) X has a proper distribution F supported on $[0, \infty)$, that is its tail distribution function $\overline{F}(x) =$

*Research supported by the National Natural Science Foundation of China (No.11071182), the National Natural Science Foundation of China (No.11401415), the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No.13KJB110025).

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$P(X > x), x \in (-\infty, \infty)$ is always positive. By definition, a distribution F is said to be heavy-tailed, if for all $\alpha > 0$,

$$\int_0^\infty e^{\alpha y} dF(y) = \infty.$$

Otherwise, F is called light-tailed. As is known to all, heavy-tailed distributions have important applications in various fields of applied probability, such as risk theory, queuing system, warehousing management, branching theory, communication net and infinite divisible distribution theory, see, for example, Embrechts et al. (1997) and Foss et al. (2013). Further, some common heavy tailed distributions are also studied and applied in the statistics, see Zeller et al. (2012), Tavangar and Hashemi (2013), Sultan and Al-Moisheer (2013), and so on. So they attract much interest of the researchers. However, the heavy-tailed distribution class is too large and it contains some distributions which can not be “dominated”, so some subclasses of the heavy-tailed distribution class with good properties were introduced. Here we first recall some existing subclasses of the heavy-tailed distribution class, and then introduce some new ones.

We say that a distribution F belongs to the subexponential distribution class, denoted by $F \in \mathcal{S}$, if

$$\overline{F^{*2}}(x) \sim 2\overline{F}(x),$$

that is $\lim \overline{F^{*2}}(x)(\overline{F}(x))^{-1} = 2$, here and after, all limits refer to x tending to infinity, unless otherwise stated. The subexponential distribution class was introduced by Chistyakov (1964) in the study of the branching process, where it was proved that the subexponential distribution class is contained in the following heavy-tailed distribution subclass. We say that a distribution F belongs to the long-tailed distribution class, denoted by $F \in \mathcal{L}$, if for any $y \in (-\infty, \infty)$,

$$\overline{F}(x+y) \sim \overline{F}(x).$$

For convenience, we write

$$C_*(F) = \liminf \overline{F^{*2}}(x)(\overline{F}(x))^{-1} \text{ and } C^*(F) = \limsup \overline{F^{*2}}(x)(\overline{F}(x))^{-1}.$$

It was obtained by Theorem 1 of Foss and Korshunov (2007) that for any heavy-tailed distribution F , $C_*(F) = 2$. And it is obvious that $F \in \mathcal{S}$ if and only if $C_*(F) = C^*(F) = 2$, which means that for a subexponential distribution F , compared with other distributions, the fluctuations of the ratios $\overline{F^{*2}}(x)(\overline{F}(x))^{-1}$ is minimal as $x \rightarrow \infty$. So to some extent, we may regard such distribution F as “optimal”. Also, we may say a distribution F is “controllable” if

$$2 \leq C^*(F) < \infty.$$

And when $C^*(F) = \infty$, we say that the distribution F is “uncontrollable”.

In a probabilistic model with heavy-tailed distributions, if we may choose distributions freely, then the first choice is of course the subexponential distributions. However, due to the complexity of the real world, the distributions usually are not decided by us. So it is necessary to study “controllable” distributions, even “uncontrollable” distributions.

Kluppelberg (1990) first considered the “controllable” distributions and called them “weak idempotents”. Shimura and Watanabe (2005) called the distributions generalized subexponential and denoted the class of such distributions by \mathcal{OS} . In the terminology of

Bingham et al. (1987), the distributions from the class \mathcal{OS} are called O-regularly varying functions. Here we continue to use the notation \mathcal{OS} .

The distribution class \mathcal{OS} is a rather large class which contains many heavy-tailed and light-tailed distributions. For research on \mathcal{OS} , besides the above-mentioned literature, the reader can refer to Klüppelberg and Villasenor (1991), Watanabe and Yamamura (2010), Lin and Wang (2012), Cheng and Wang (2012), Cheng et al. (2012), Yu and Wang (2013), Gao et al. (2013), Beck et al. (2013), Xu et al. (2015a,b,c), and so on.

In present paper, we are interested in some subclasses of the class \mathcal{OS} , which respectively correspond two subclasses of the class \mathcal{S} as follows.

We say that a distribution F belongs to the strong subexponential distribution class, denoted by $F \in \mathcal{S}^*$, if $0 < EX < \infty$ and

$$\int_0^x \bar{F}(x-y)\bar{F}(y)dy \sim 2EX\bar{F}(x).$$

In the above formulas, if the distribution F is supported on $(-\infty, \infty)$, then EX is replaced by EX^+ , where $X^+ = X\mathbf{1}(X \geq 0)$.

We say that a distribution F belongs to the locally long-tailed distribution class, denoted by $F \in \mathcal{L}_{\Delta_T}$, where T is some positive constant or ∞ , if for some constant $x_0 > 0$, $F(x + \Delta_T) = P(X \in x + \Delta_T) > 0$ for all $x \geq x_0$ and the relationship

$$F(x + y + \Delta_T) \sim F(x + \Delta_T)$$

holds uniformly for all $y \in (0, T]$, where $\Delta_T = (0, T]$, $x + \Delta_T = (x, x + T]$ when $T < \infty$, and $\Delta_T = (0, \infty)$, $x + \Delta_T = (x, \infty)$ when $T = \infty$. Further, we say that a distribution F belongs to the locally subexponential distribution class, denoted by $F \in \mathcal{S}_{\Delta_T}$, if for some $0 < T \leq \infty$, $F \in \mathcal{L}_{\Delta_T}$ and

$$F^{*2}(x + \Delta_T) \sim 2F(x + \Delta_T).$$

The strong subexponential distribution and the locally subexponential distribution were introduced by Klüppelberg (1988) and Asmussen et al. (2003), respectively. Inspired by the distribution classes \mathcal{S}^* , \mathcal{S}_{Δ_T} and \mathcal{OS} , we introduce the following two new distribution classes which are the main object of study of present paper.

Definition 1.1. We say that a distribution F belongs to the generalized strong subexponential distribution class, denoted by $F \in \mathcal{OS}^*$, if

$$C^\otimes(F) = \limsup \int_0^x \bar{F}(x-y)\bar{F}(y)dy (\bar{F}(x))^{-1} < \infty.$$

Definition 1.2. We say that a distribution F belongs to the generalized locally subexponential distribution class for some $0 < T \leq \infty$, denoted by $F \in \mathcal{OS}_{\Delta_T}$, if for some constant $x_0 > 0$, $F(x + \Delta_T) > 0$ for all $x \geq x_0$ and

$$C^T(F) = \limsup F^{*2}(x + \Delta_T) (F(x + \Delta_T))^{-1} < \infty.$$

Obviously, like the class \mathcal{OS} , the classes \mathcal{OS}^* and \mathcal{OS}_{Δ_T} contain many heavy-tailed distributions and light-tailed distributions. Moreover, the classes \mathcal{OS}^* and \mathcal{OS}_{Δ_T} have

also certain “controllability”. Existing research and application on class \mathcal{OS}^* , for example, can be found in Proposition 1.1, 1.2, 1.4 and Lemma 2.1 of Xu et al. (2015b).

Similar to $C_*(F)$, we write

$$C_{\otimes}(F) = \liminf \int_0^x \overline{F}(x-y)\overline{F}(y)dy(\overline{F}(x))^{-1}$$

and

$$C_T(F) = \liminf F^{2*}(x + \Delta_T)(F(x + \Delta_T))^{-1}.$$

For a heavy-tailed distribution F , apart from proving the fact that $C_*(F) = 2$, Foss and Korshunov (2007) also proved that $C_{\otimes}(F) = 2EX$. However, for a light-tailed distribution F , the equalities $C_{\otimes}(F) = 2EX$ do not necessarily hold, see Foss and Korshunov (2007). Similarly, for a locally heavy-tailed distribution F , the equality $C_T(F) = 2$ for some $0 < T < \infty$ also does not necessarily hold, but if $F \in \mathcal{L}_{\Delta_T}$ for some $0 < T < \infty$, then $C_T(F) = 2$, see Proposition 4.1 and Remark 4.1 of Chen et al. (2013).

The remainder of this paper consists of four sections. In Section 2, the relationships among the two new distribution classes and some existing related ones are discussed. Some examples of long-tailed distribution show that the class \mathcal{OS}^* and the class \mathcal{OS}_{Δ_T} properly contain the class \mathcal{S}^* and the class \mathcal{S}_{Δ_T} , respectively. It should be said that the methods of construction of these distributions are not trivial, but these distributions are not particularly weird, especially their integral tail distributions are much more normal. And proofs of this examples are given in the Section 5. In Section 3, the local asymptotic estimation for the supremum of a random walk is presented, where the summands of the random walk have common long-tailed and generalized strong subexponential distribution. To this end, we find out some relationship between the random walk with heavy-tailed summands and the random walk with light-tailed summands. Some applications of the above results are given in Section 4.

2 The relationships among the distribution classes

2.1 The relation between the classes $\mathcal{L} \cap \mathcal{OS}^*$ and \mathcal{S}^* .

Proposition 2.1. *The inclusion relation $\mathcal{S}^* \subset \mathcal{L} \cap \mathcal{OS}^*$ is proper.*

Proof. Obviously, the inclusion relation $\mathcal{S}^* \subset \mathcal{L} \cap \mathcal{OS}^*$ holds. So, we just prove that the relationship is proper through the following two types of distributions.

Example 2.1. *Let $m \geq 1$ be any integer. Choose any constant $\alpha \in (m^{-1}, 1 + m^{-1})$ and any constant $x_1 > 4^{m\alpha(m\alpha-1)^{-1}}$. For all integers $n \geq 1$, let $x_{n+1} = x_n^{2-(m\alpha)^{-1}}$. Clearly, $x_{n+1} > 4x_n$ and $x_n \rightarrow \infty$ as $n \rightarrow \infty$. Now, define the distribution F as follows:*

$$\begin{aligned} \overline{F}(x) = & \mathbf{1}(x < 0) + (x_1^{-1}(x_1^{-\alpha} - 1)x + 1)\mathbf{1}(0 \leq x < x_1) \\ & + \sum_{n=1}^{\infty} \left((x_n^{-\alpha} + (x_n^{-2\alpha-1+m^{-1}} - x_n^{-\alpha-1})(x - x_n))\mathbf{1}(x_n \leq x < 2x_n) \right. \\ & \left. + x_n^{-2\alpha+m^{-1}}\mathbf{1}(2x_n \leq x < x_{n+1}) \right), \quad x \in (-\infty, \infty). \end{aligned} \quad (2.1)$$

Further, let

$$\overline{G_m}(x) = (\overline{F}(x))^m = \overline{F}^m(x), \quad x \in (-\infty, \infty).$$

Then $G_m \in (\mathcal{S} \cap \mathcal{OS}^*) \setminus \mathcal{S}^*$.

Example 2.2. Let $m \geq 1$ be any integer. Choose any constant $\alpha \in (2 + 2m^{-1}, \infty)$ and any constant $x_1 > 4^\alpha$. And, for all integers $n \geq 1$, let $x_{n+1} = x_n^{1+\alpha^{-1}}$. Clearly, $x_{n+1} > 4x_n$ and $x_n \rightarrow \infty$ as $n \rightarrow \infty$. Now, define the distribution F as follows:

$$\begin{aligned} \overline{F}(x) = & \mathbf{1}(x < 0) + (x_1^{-1}(x_1^{-\alpha} - 1)x^{2^{-1}} + 1)\mathbf{1}(0 \leq x < x_1^2) \\ & + \sum_{n=1}^{\infty} \left((x_n^{-\alpha} + (x_n^{-\alpha-2} - x_n^{-\alpha-1})(x^{2^{-1}} - x_n))\mathbf{1}(x_n^2 \leq x < 4x_n^2) \right. \\ & \left. + x_n^{-\alpha-1}\mathbf{1}(4x_n^2 \leq x < x_{n+1}^2) \right), \quad x \in (-\infty, \infty). \end{aligned} \quad (2.2)$$

Further, let G_m be the same as in Example 2.1, then $G_m \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}$, thus $G_m \notin \mathcal{S}^*$.

Therefore, the proposition is proved. \square

2.2 The relation between the classes $\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ and \mathcal{S}_{Δ_T}

Proposition 2.2. For all $0 < T \leq \infty$, the inclusion relation $\mathcal{S}_{\Delta_T} \subset \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T} \subset \mathcal{L}_{\Delta_T} \cap \mathcal{OS}$ is proper.

Proof. When $T = \infty$, the corresponding counterexamples showing that there exist some distributions belonging to the class $\mathcal{L} \cap \mathcal{OS} \setminus \mathcal{S}$ may be found in Leslie (1989), Lin and Wang (2012), Example 2.2 and Example 2.5 below. So we only prove the result in the case that $0 < T < \infty$. First, we prove a simple fact that $\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T} \subset \mathcal{L}_{\Delta_T} \cap \mathcal{OS}$. Let $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for some $0 < T < \infty$, we have

$$\begin{aligned} \int_0^x \overline{V}(x-y)dV(y) & \leq \sum_{k=0}^{[xT^{-1}]} \sum_{l=0}^{\infty} V(x+lT - (k+1)T + \Delta_T)V(kT + \Delta_T) \\ & = O\left(\sum_{l=0}^{\infty} \sum_{k=0}^{[xT^{-1}]} \int_{kT}^{(k+1)T} V(x+lT - y + \Delta_T)dV(y)\right) \\ & = O\left(\sum_{l=0}^{\infty} \int_0^x V(x+lT - y + \Delta_T)dV(y)\right) = O(\overline{V}(x)), \end{aligned}$$

thus $V \in \mathcal{OS}$, where $a(x) = O(b(x))$ mean that $\limsup a(x)(b(x))^{-1} \leq 1$ for two positive functions a and b .

The following Example 2.5 shows that the distribution class $\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ is properly included in the distribution class $\mathcal{L}_{\Delta_T} \cap \mathcal{OS}$. Now, we give three counterexamples to show that the inclusion relationship $\mathcal{S}_{\Delta_T} \subset \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ is also proper. To this end, we first introduce a concept of distribution. For some distribution F with a finite and positive mean EX , we say that the distribution F^I defined by

$$F^I(x) = (EX)^{-1} \int_0^x \overline{F}(y)dy \mathbf{1}(x > 0), \quad x \in (-\infty, \infty)$$

is the integrated tail distribution (or equilibrium distribution) of the distribution F . Related work on the integrated tail distribution can be found in Klüppelberg (1988), Korshunov (1997), Li and Xu (2008), and so on.

Example 2.3. For any $m \geq 1$, let G_m be the same as in Example 2.1 or Example 2.2. Then for all $0 < T < \infty$, $G_m^I \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}$.

In order to give the third counterexample, we first introduce some relevant notions and notations. We say that a distribution F belongs to the exponential distribution class with the index $\gamma \geq 0$, denoted by $F \in \mathcal{L}(\gamma)$, if for all $t \in (-\infty, \infty)$,

$$\overline{F}(x+t) \sim e^{-\gamma t} \overline{F}(x).$$

We say that a distribution F belongs to the convolution equivalent distribution class with the index $\gamma \geq 0$, denoted by $F \in \mathcal{S}(\gamma)$, if $F \in \mathcal{L}(\gamma)$, $M_\gamma(F) = \int_0^\infty e^{\gamma y} dF(y) < \infty$ and

$$\overline{F^{*2}}(x) \sim 2M_\gamma(F) \overline{F}(x).$$

Obviously, when $\gamma = 0$, $\mathcal{L}(0) = \mathcal{L}$ and $\mathcal{S}(0) = \mathcal{S}$; when $\gamma > 0$, the distributions in $\mathcal{L}(\gamma)$ are light-tailed. The classes $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$ were introduced by Chover et al. (1973, a, b) for $\gamma > 0$. Bertoin and Doney (1996) note that, however, in definitions of the class $\mathcal{L}(\gamma)$ and the class $\mathcal{S}(\gamma)$, if $\gamma > 0$, and if the distribution F is lattice, then x and T should be restricted to values of the lattice span.

Further, for a distribution F , if $M_\gamma(F) < \infty$ for some $\gamma > 0$, we may define a new distribution as follows.

$$F_\gamma(x) = (M_\gamma(F))^{-1} \int_0^x e^{\gamma y} dF(y) \mathbf{1}(x \geq 0), \quad x \in (-\infty, \infty),$$

which is called the γ -transform or the Escher transform of the distribution F . Similarly, we can define the $-\gamma$ -transform of a distribution F for any $\gamma > 0$.

Example 2.4. Klüppelberg and Villasenor (1991) found two distributions $F_i \in \mathcal{S}(\gamma)$ for some $\gamma > 0$, $i = 1, 2$, but $F = F_1 * F_2 \in \mathcal{L}(\gamma) \setminus \mathcal{S}(\gamma)$. Then for all $0 < T < \infty$, $F_\gamma \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}$. \square

Remark 2.1. We point out that the distribution $F_\gamma \in \mathcal{L}_{\Delta_T} \setminus \mathcal{S}_{\Delta_T}$ for all $0 < T < \infty$ in Example 2.4 was firstly introduced by Proposition 2.1 of Chen et al. (2013). In addition, there Example 2.3 and Example 2.4 give two new ways to find more distributions in the class $(\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$ and its subclasses.

2.3 The relation between the classes $\mathcal{L} \cap \mathcal{OS}$ and $\mathcal{L} \cap \mathcal{OS}^*$.

Proposition 2.3. The inclusion relation $\mathcal{L} \cap \mathcal{OS}^* \subset \mathcal{L} \cap \mathcal{OS}$ is proper.

Proof. First, using the method of Lemma 9 of Denisov et al. (2004), we can prove the fact that $\mathcal{L} \cap \mathcal{OS}^* \subset \mathcal{L} \cap \mathcal{OS}$. Next, we prove that the above inclusion relation is proper by using the following example. To this end, we recall a distribution in the distribution class $\mathcal{L} \cap \mathcal{OS} \setminus \mathcal{S}$, which was found by Lin and Wang (2012).

Example 2.5. Let $x_1 > 1$ be any given number, and let $x_{n+1} = (2x_n)^2$, $n \geq 1$. For any $\alpha \in (0, 1)$, define

$$\begin{aligned}\overline{F}(x) &= \mathbf{1}(x < 0) + (x_1^{-1}(x_1^{-\alpha} - 1)x + 1)\mathbf{1}(0 \leq x < x_1) \\ &\quad + \sum_{n=1}^{\infty} \left((x_n^{-\alpha} + (2^{-2\alpha}x_n^{-2\alpha-1} - x_n^{-\alpha-1})(x - x_n))\mathbf{1}(x_n \leq x < 2x_n) \right. \\ &\quad \left. + (2x_n)^{-2\alpha}\mathbf{1}(2x_n \leq x < x_{n+1}) \right), \quad x \in (-\infty, \infty).\end{aligned}\tag{2.3}$$

For any positive integer $m \in (\alpha^{-1}, 2\alpha^{-1})$, let $\overline{G}_m(x) = \overline{F}^m(x)$, $x \in (-\infty, \infty)$. It is obvious that the distribution G_m has a finite mean. Lin and Wang (2012) has proved that $G_m \in (\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$. Further, we have $G_m \notin \mathcal{OS}^*$, $G_m^I \in (\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$ and $G_m^I \in \mathcal{L}_{\Delta_T} \setminus \mathcal{OS}_{\Delta_T}$ for all $0 < T < \infty$. \square

3 Local asymptotic estimations

In this section, we try to deliver local asymptotic estimations for the supremum of a random walk, where the distributions of the summands of the random walk belong to the class $\mathcal{L} \cap \mathcal{OS}^*$, or equivalently, the integrated tail distributions of the summands belong to the class $\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for any $0 < T < \infty$, see Lemma 3.5 below. This explains that distributions from the classes $\mathcal{L} \cap \mathcal{OS}^*$ and $\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ possess good properties, thus they have important value in applications.

In the following, we first introduce some concepts of a random walk and the main result of this paper. Then we give some lemmas in the second subsection. The proof of the main result will be presented at last.

3.1 Related concepts and main result

Let $\{X_i : i \geq 1\}$ be a sequence of independent, identically distributed r.v.s with a common non-degenerate distribution F supported on $(-\infty, \infty)$. Denote the random walk by $\{S_n = \sum_{i=1}^n X_i : n \geq 0\}$, where $S_0 = 0$, and the supremum of the random walk by $M = \sup_{n \geq 0} S_n$ with a distribution W supported on $[0, \infty)$. Assume that

$$-\infty < -\mu = EX_1 < 0,$$

then we know that S_n drifts to $-\infty$ and W is a proper distribution.

Further, let $\tau_+ = \inf\{n \geq 1 : S_n > 0\}$ be the first ascending ladder-epoch and S_{τ_+} the first ascending ladder height with a defective distribution F_+ supported on $[0, \infty)$, i.e., $0 < p = F_+(\infty) < 1$. Denote $G(x) = p^{-1}F_+(x)$, $x \in (-\infty, \infty)$, then G is a proper distribution supported on $[0, \infty)$. It is well known that, for any $0 < T \leq \infty$ and $x \geq 0$,

$$W(x + \Delta_T) = (1 - p) \sum_{n=1}^{\infty} p^n G^{*n}(x + \Delta_T),\tag{3.1}$$

see Asmussen et al. (2002) or Asmussen et al. (2003).

For the random walk $\{S_n : n \geq 0\}$, if $F \in \mathcal{S}^*$, then

$$W(x + \Delta_T) \sim \mu^{-1} T \bar{F}(x), \quad (3.2)$$

for any $0 < T < \infty$, see Asmussen et al. (2002), and so on. Further, Asmussen et al. (2003) show that $W \in \mathcal{S}_{\Delta_T}$ for any $0 < T < \infty$. Naturally, one hopes to know that if $F \in \mathcal{L} \cap \mathcal{OS}^*$, then how to estimate $W(x + \Delta_T)$? And, what is distribution of the supremum M ? Our answer is as follows.

Theorem 3.1. *For the random walk $\{S_n : n \geq 0\}$, if $F \in \mathcal{L}$, then for any $0 < T < \infty$,*

$$\liminf W(x + \Delta_T) (\bar{F}(x))^{-1} = \mu^{-1} T. \quad (3.3)$$

Further, if $F \in \mathcal{L} \cap \mathcal{OS}^$ and*

$$C^\otimes(F) < \mu + 2EX_1^+, \quad (3.4)$$

then for any $0 < T < \infty$,

$$\limsup W(x + \Delta_T) (\bar{F}(x))^{-1} \leq \mu^{-1} T (1 - \mu^{-1} (C^\otimes(F) - 2EX_1^+))^{-1}, \quad (3.5)$$

and $W \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$. In particular, if $C^\otimes(F) = 2EX_1^+$, namely $F \in \mathcal{S}^$, then (3.2) holds. In addition, if $F \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}^*$, then $W \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}$.*

Remark 3.1. *Here we note that the theorem gives also us a new way to find more distributions in the class $(\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}$ for any $0 < T < \infty$.*

In addition, we give the following results to illustrate the condition (3.4)

Proposition 3.4. *There exists a long-tailed and generalized strong subexponential distribution F supported on $(-\infty, \infty)$ satisfying the condition (3.4).*

3.2 Some lemmas

In this section, we prepare more lemmas on the local distributions, which will be used in the proof of Theorem 3.1 and also have their own independent value.

First, we recall a known fact. If a distribution $V \in \mathcal{L}_{\Delta_T}$ for some $0 < T \leq \infty$, then

$$\begin{aligned} \mathcal{H}_{\Delta_T}(V) = & \{h \text{ on } [0, \infty) : h(x) \uparrow \infty, h(x) = o(x) \text{ and} \\ & V(x + t + \Delta_T) \sim V(x + \Delta_T) \text{ holds uniformly for all } |t| \leq h(x)\} \neq \emptyset. \end{aligned}$$

And if $h \in \mathcal{H}_{\Delta_T}(V)$ and $h(x) \geq h_1(x) \uparrow \infty$, then $h_1 \in \mathcal{H}_{\Delta_T}(V)$ too. Particularly, when $T = \infty$, we denote $\mathcal{H}_{\Delta_\infty}(V)$ by $\mathcal{H}(V)$.

Lemma 3.1. (i) *If $V \in \mathcal{L}_{\Delta_T}$ for some $0 < T \leq \infty$, and for some $h \in \mathcal{H}_{\Delta_T}(V)$,*

$$\int_{h(x)}^{x-h(x)} V(x - y + \Delta_T) dV(y) = O(V(x + \Delta_T)), \quad (3.6)$$

then $V \in \mathcal{OS}_{\Delta_T}$.

(ii) *If $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$, then for all $h \in \mathcal{H}_{\Delta_T}(V)$, (3.6) holds and*

$$\limsup \int_{h(x)}^{x-h(x)} V(x - y + \Delta_T) (V(x + \Delta_T))^{-1} dV(y) = C^T(V) - 2. \quad (3.7)$$

Proof. By $V \in \mathcal{L}_{\Delta_T}$ and a standard method, we have

$$V^{*2}(x + \Delta_T) \sim 2V(x + \Delta_T) + \int_{h(x)}^{x-h(x)} V(x - y + \Delta_T) dV(y), \quad (3.8)$$

thus $V \in \mathcal{OS}_{\Delta_T}$ follows immediately from (3.8) and (3.6).

On the other hand, if $V \in \mathcal{OS}_{\Delta_T}$, then (3.6) and (3.7) follow directly from (3.8). \square

Lemma 3.2. *If $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for some $0 < T \leq \infty$, then for all $n \geq 1$,*

$$\limsup V^{*n}(x + \Delta_T)(V(x + \Delta_T))^{-1} \leq \sum_{k=0}^{n-1} (C^T(V) - 1)^{n-1-k}. \quad (3.9)$$

Proof. Apparently, (3.9) holds for $n = 1, 2$. We assume that (3.9) holds for $n = m$ and aim to show that it holds for $n = m + 1$ too. For any $n \geq 1$, denote

$$C_n^T(V) = \limsup V^{*n}(x + \Delta_T)(V(x + \Delta_T))^{-1}.$$

For any $h \in \mathcal{H}_{\Delta_T}(V) \cap \mathcal{H}_{\Delta_T}(V^{*m})$, by a standard method, we obtain

$$V^{*(m+1)}(x + \Delta_T) \sim V(x + \Delta_T) + V^{*m}(x + \Delta_T) + I(x), \quad (3.10)$$

where

$$\begin{aligned} I(x) &\leq \int_{h(x)-T}^{x-h(x)+T} V^{*m}(x - y + \Delta_T) dV(y) \\ &\lesssim C_m^T(V) \int_{h(x)-T}^{x-h(x)+T} V(x - y + \Delta_T) dV(y). \end{aligned} \quad (3.11)$$

It follows from (3.9)-(3.11) and Lemma 3.1 that

$$\begin{aligned} C_{m+1}^T(V) &\leq C_m^T(V)(C^T(V) - 1) + 1 \\ &\leq \sum_{k=0}^m (C^T(V) - 1)^{m-k}, \end{aligned}$$

namely (3.9) holds for $n = m + 1$. \square

Lemma 3.3. *Let V be a proper distribution on $[0, \infty)$. If $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for some $0 < T \leq \infty$, then for arbitrary $\varepsilon > 0$, there exists $x_1 > 0$ and $K = K(\varepsilon, x_1) > 0$ such that for all $x \geq x_1$ and $n \geq 1$,*

$$V^{*n}(x + \Delta_T) \leq K(C^T(V) - 1 + \varepsilon)^n V(x + \Delta_T). \quad (3.12)$$

Proof. For any $h \in \mathcal{H}_{\Delta_T}(V)$, by the condition $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$, we know that for any $\varepsilon \in (0, 1)$, there exists a positive number x_1 large enough such that when $x \geq x_1$, $h(x) > 2T$,

$$|V(x - y + \Delta_T)(V(x + \Delta_T))^{-1} - 1| < 8^{-1}\varepsilon \quad \text{uniformly for } |y| \leq h(x) \quad (3.13)$$

and

$$\int_{h(x)-T}^{x-h(x)+T} V(x-y+\Delta_T) dV(y) < (C^T(V) - 2 + 8^{-1}\varepsilon) V(x+\Delta_T). \quad (3.14)$$

We now prove the lemma by induction. When $n = 1$, (3.12) is obvious. Assume that (3.12) holds for a fixed integer $n \geq 1$, we now show that it holds for $n + 1$. Denote

$$A_n = \sup_{x \geq x_1} V^{*n}(x + \Delta_T)(V(x + \Delta_T))^{-1}, \quad n \geq 1.$$

We have

$$\begin{aligned} V^{*(n+1)}(x + \Delta_T) &= \int_0^{h(x)} V^{*n}(x - y + \Delta_T) dV(y) + \int_0^{h(x)} V(x - y + \Delta_T) dV^{*n}(y) \\ &\quad + P(S_{n+1} \in x + \Delta_T, S_n > h(x), X_{n+1} > h(x)) \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned} \quad (3.15)$$

When $x \geq x_1$, by (3.13), we have

$$\begin{aligned} I_1(x)(V(x + \Delta_T))^{-1} &= \int_0^{h(x)} V^{*n}(x - y + \Delta_T)(V(x + \Delta_T))^{-1} dV(y) \\ &\leq A_n \int_0^{h(x)} V(x - y + \Delta_T)(V(x + \Delta_T))^{-1} dV(y) \\ &\leq A_n (1 + 8^{-1}\varepsilon). \end{aligned} \quad (3.16)$$

Similarly, we have

$$I_2(x)(V(x + \Delta_T))^{-1} \leq 1 + 8^{-1}\varepsilon. \quad (3.17)$$

Finally, by (3.14),

$$\begin{aligned} I_3(x)(V(x + \Delta_T))^{-1} &\leq \int_{h(x)-T}^{x-h(x)+T} V^{*n}(x - y + \Delta_T)(V(x + \Delta_T))^{-1} dV(y) \\ &\leq A_n \int_{h(x)-T}^{x-h(x)+T} V(x - y + \Delta_T)(V(x + \Delta_T))^{-1} dV(y) \\ &\leq A_n (C^T(V) - 2 + 8^{-1}\varepsilon). \end{aligned} \quad (3.18)$$

So when $x \geq x_1$, it follows from (3.15)-(3.18) that

$$\begin{aligned} A_{n+1} &\leq \sup_{x \geq x_1} V^{*n}(x + \Delta_T)(V(x + \Delta_T))^{-1} \\ &\leq 1 + 8^{-1}\varepsilon + A_n (C^T(V) - 1 + 4^{-1}\varepsilon). \end{aligned} \quad (3.19)$$

Taking $K = K(\varepsilon) =: \frac{8}{3\varepsilon}$ and using (3.19), we get

$$\begin{aligned} A_{n+1} &\leq 2 + K (C^T(V) - 1 + 3 \cdot 4^{-1}\varepsilon)^{n+1} \\ &\leq K (C^T(V) - 1 + \varepsilon)^{n+1}. \end{aligned}$$

This completes the proof of Lemma 3.3. \square

For a r.v. ξ with a distribution V and a positive and finite mean $E\xi$, denote

$$V_1(x) = \min\{1, V^I(x)E\xi\}, \quad x \in (-\infty, \infty),$$

which may be a defective distribution.

Lemma 3.4. *Let two distributions V and V_1 be described as above. If $V \in \mathcal{L}$, then $V_1 \in \mathcal{L}_{\Delta_T}$ for all $0 < T < \infty$; on the contrary, if $V_1 \in \mathcal{L}_{\Delta_T}$ for some $0 < T < \infty$, then $V \in \mathcal{L}$. And both are able to derive the following asymptotic equivalence formula:*

$$V_1(x + \Delta_T) \sim \bar{V}(x)T. \quad (3.20)$$

Proof. Clearly, since $V \in \mathcal{L}$, $V_1 \in \mathcal{L}_{\Delta_T}$ for all $0 < T < \infty$ and (3.20) holds. On the contrary, from the following fact that for any $y > 0$ and x large enough,

$$\bar{V}(x - y)T \leq V_1(x - y - T + \Delta_T) \sim V_1(x + \Delta_T) \leq \bar{V}(x)T,$$

we know that $V \in \mathcal{L}$ and (3.20) holds. \square

In the following, we give some new versions of Pitman's theorem. When $T = \infty$, the result is due to Pitman (1980). To this end, we first recall a known fact. Yu and Wang (2014) show that, if a distribution $V \in \mathcal{L} \cap \mathcal{OS}$, then for any $h \in \mathcal{H}(V)$,

$$\limsup \int_{h(x)}^{x-h(x)} \bar{V}(x - y)(\bar{V}(x))^{-1} dV(y) = C^*(V) - 2.$$

And by Lemma 3.1 (ii), if $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for some $0 < T \leq \infty$, then (3.7) holds. Similar to the proof of (3.7), we can prove that, if a r.v. ξ with distribution $V \in \mathcal{L} \cap \mathcal{OS}^*$, then for any $h \in \mathcal{H}(V)$,

$$\limsup \int_{h(x)}^{x-h(x)} \bar{V}(x - y)\bar{V}(y)(\bar{V}(x))^{-1} dy = C^\otimes(V) - 2E\xi. \quad (3.21)$$

Lemma 3.5. *Let V and U be two distributions, then the following assertions hold.*

(i) *If $V \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for some $0 < T \leq \infty$ and there exist two constants c_1 and c_2 such that*

$$0 < c_1 = \liminf \frac{U(x + \Delta_T)}{V(x + \Delta_T)} \leq \limsup \frac{U(x + \Delta_T)}{V(x + \Delta_T)} = c_2 < \infty, \quad (3.22)$$

then $U \in \mathcal{OS}_{\Delta_T}$ and

$$C^T(U) - 2c_1^{-1}c_2 \leq c_1^{-1}c_2^2(C^T(V) - 2). \quad (3.23)$$

Particularly, if $c_1 = c_2 = c_0$, then $U \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ and

$$C^T(U) - 2 = c_0(C^T(V) - 2). \quad (3.24)$$

(ii) *If $V \in \mathcal{L} \cap \mathcal{OS}^*$ and there exist two constants c_1 and c_2 such that*

$$-\infty < c_1 = \liminf U(x + \Delta_T)(\bar{V}(x))^{-1} \leq \limsup U(x + \Delta_T)(\bar{V}(x))^{-1} = c_2 < \infty \quad (3.25)$$

for some $0 < T < \infty$, then $U \in \mathcal{OS}_{\Delta_T}$ and

$$c_1^2(c_2T)^{-1}(C_{\otimes}(V) - 2E\xi) \leq C_T(U) - 2 \leq C^T(U) - 2 \leq c_2^2(c_1T)^{-1}(C^{\otimes}(V) - 2E\xi). \quad (3.26)$$

Particularly, if $c_1 = c_2 = c_0$, then both are able to derive the following equations:

$$C^T(U) - 2 = c_0T^{-1}(C^{\otimes}(V) - 2E\xi) \text{ and } C_T(U) - 2 = c_0T^{-1}(C_{\otimes}(V) - 2E\xi). \quad (3.27)$$

Thus, $V \in \mathcal{L} \cap \mathcal{OS}^*$ if and only if $U \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for all $0 < T < \infty$. Further, we have

$$C^T(V_1) - 2 = C^{\otimes}(V) - 2E\xi \text{ and } C_T(V_1) - 2 = c_0T^{-1}(C_{\otimes}(V) - 2E\xi). \quad (3.28)$$

Proof. (i) In (3.15), we take $n = 1$ and $V = U$, then for any $h \in \mathcal{H}_{\Delta_T}(V)$,

$$\limsup I_i(x)(U(x + \Delta_T))^{-1} \leq c_1^{-1}c_2, \quad i = 1, 2. \quad (3.29)$$

Let r.v.s ξ_1 and ξ_2 have distributions V and U respectively. Then by (3.22), we have

$$\begin{aligned} & \limsup \frac{I_3(x)}{U(x + \Delta_T)} \leq c_2 \limsup \int_{h(x)-T}^{x-h(x)+T} \frac{V(x-y+\Delta_T)}{U(x+\Delta_T)} dU(y) \\ &= c_2 \limsup P(\xi_1 + \xi_2 \in x + \Delta_T, h(x) - T < \xi_2 \leq x - h(x) + T)(U(x + \Delta_T))^{-1} \\ &\leq c_2 \limsup P(\xi_1 + \xi_2 \in x + \Delta_T, h(x) - 2T < \xi_1 \leq x - h(x) + 2T)(U(x + \Delta_T))^{-1} \\ &= c_2 \limsup \int_{h(x)-2T}^{x-h(x)+2T} U(x-y+\Delta_T)(U(x+\Delta_T))^{-1} dV(y) \\ &\leq c_1^{-1}c_2^2 \limsup \int_{h(x)-2T}^{x-h(x)+2T} V(x-y+\Delta_T)(V(x+\Delta_T))^{-1} dV(y) \\ &= c_1^{-1}c_2^2(C^T(V) - 2). \end{aligned}$$

Thus, by (3.8) of Lemma 3.1, (3.29) and (3.7), we know that $U \in \mathcal{OS}_{\Delta_T}$ and (3.23) holds.

Particularly, if $c_1 = c_2 = c_0$, then

$$C^T(V) - 2 \leq c_0^{-1}(C^T(U) - 2).$$

Combining the above inequalities and (3.23) yields the equality (3.24).

(ii) When $V \in \mathcal{L} \cap \mathcal{OS}^*$, similarly to (i), for some $h \in \mathcal{H}_{\Delta_T}(U)$, we have

$$\limsup I_i(x)(U(x + \Delta_T))^{-1} \leq c_1^{-1}c_2, \quad i = 1, 2. \quad (3.30)$$

For $I_3(x)$, without loss of generality, we may assume that $l_1(x) = (x - 2h_1(x))T^{-1}$ is an integer for x large enough, where $h_1(x) = h(x) - T$. Thus, by (3.22), we have

$$\begin{aligned} & \limsup I_3(x)(U(x + \Delta_T))^{-1} \leq c_2 \limsup \int_{h_1(x)}^{x-h_1(x)} \bar{V}(x-y)(U(x + \Delta_T))^{-1} dU(y) \\ &= c_2 \limsup \sum_{k=1}^{l_1(x)} \int_{h_1(x)+(k-1)T}^{h_1(x)+kT} \bar{V}(x-y)(U(x + \Delta_T))^{-1} dU(y) \\ &\leq c_1^{-1}c_2^2 \limsup \sum_{k=1}^{l_1(x)} \bar{V}(x-h_1(x)-kT)\bar{V}(h_1(x)+(k-1)T)(\bar{V}(x))^{-1} \\ &= c_1^{-1}c_2^2T^{-1} \limsup \int_{h_1(x)}^{x-h_1(x)} \bar{V}(x-y)\bar{V}(y)(\bar{V}(x))^{-1} dy \\ &= c_1^{-1}c_2^2T^{-1}(C^{\otimes}(V) - 2E\xi). \end{aligned}$$

Thus, $U \in \mathcal{OS}_{\Delta_T}$.

On the other hand, we have

$$\liminf I_i(x)(U(x + \Delta_T))^{-1} \geq c_1 c_2^{-1}, \quad i = 1, 2. \quad (3.31)$$

And without loss of generality, we may assume that $l(x) = (x - 2h(x))T^{-1}$ is an integer for x large enough, then

$$\begin{aligned} \liminf I_3(x)(U(x + \Delta_T))^{-1} &\geq c_1 \liminf \int_{h(x)}^{x-h(x)} \bar{V}(x-y)(U(x + \Delta_T))^{-1} dU(y) \\ &= c_1 \liminf \sum_{k=1}^{l(x)} \int_{h(x)+(k-1)T}^{h(x)+kT} \bar{V}(x-y)(U(x + \Delta_T))^{-1} dU(y) \\ &\geq c_1^2 c_2^{-1} \liminf \sum_{k=1}^{l(x)} \bar{V}(x-h(x)-(k-1)T) \bar{V}(h(x)+(k-1)T) (\bar{V}(x))^{-1} \\ &= c_1^2 c_2^{-1} T^{-1} \liminf \int_{h(x)}^{x-h(x)} \bar{V}(x-y) \bar{V}(y) (\bar{V}(x))^{-1} dy \\ &= c_1^2 c_2^{-1} T^{-1} (C_{\otimes} - 2E\xi). \end{aligned}$$

By the above four inequalities, we know that (3.26) holds.

From these results, the final result can be obtained immediately. \square

Now we discuss the relationship between a distribution V and its γ -transformation V_{γ} .

Lemma 3.6. *For some $\gamma > 0$, $V \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ if and only if $V_{\gamma} \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for any $0 < T < \infty$. And both of them imply the following asymptotic equivalence formula:*

$$C^T(V_{\gamma}) - 2 = (M_{\gamma}(V))^{-1} \gamma (C^{\otimes}(V) - 2E\xi) = (M_{\gamma}(V))^{-1} C^*(V) - 2. \quad (3.32)$$

Proof. According to Proposition 2.1 of Wang and Wang (2011), we know that $V \in \mathcal{L}(\gamma)$ for some $\gamma > 0$ if and only if $V_{\gamma} \in \mathcal{L}_{\Delta_T}$ for any $0 < T < \infty$. And both of them are able to derive the following asymptotic equivalence formula:

$$V_{\gamma}(x + \Delta_T) \sim (M_{\gamma}(V))^{-1} \gamma T e^{\gamma x} \bar{V}(x). \quad (3.33)$$

Select h and l as in the proof of Lemma 3.5. By (3.33), we know that

$$\begin{aligned} \int_{h(x)}^{x-h(x)} \frac{\bar{V}(x-y) \bar{V}(y)}{\bar{V}(x)} dy &\sim \frac{M_{\gamma}(V)}{\gamma T} \sum_{k=1}^{l(x)} \int_{h(x)+(k-1)T}^{h(x)+kT} \frac{V_{\gamma}(x-y+\Delta_T) V_{\gamma}(y+\Delta_T)}{V_{\gamma}(x+\Delta_T)} dy \\ &\sim M_{\gamma}(V) (\gamma^{-1} \sum_{k=1}^{l(x)} V_{\gamma}(x-h(x)-kT+\Delta_T) V_{\gamma}(h(x)+kT+\Delta_T) (V_{\gamma}(x+\Delta_T))^{-1} \\ &\sim M_{\gamma}(V) \gamma^{-1} \int_{h(x)}^{x-h(x)} V_{\gamma}(x-y+\Delta_T) (V_{\gamma}(x+\Delta_T))^{-1} dV_{\gamma}(y) \\ &= \gamma^{-1} \int_{h(x)}^{x-h(x)} \bar{V}(x-y) (\bar{V}(x))^{-1} dV(y). \end{aligned}$$

Thus, $V \in \mathcal{OS}$ if and only if $V_\gamma \in \mathcal{OS}_{\Delta_T}$ for any $0 < T < \infty$, and both of them are able to derive the (3.32) holds. \square

Finally, we introduce Corollary 2.1 of Yu and Wang (2014).

Lemma 3.7. *Let $V \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $\gamma \geq 0$ and $U = \sum_{n=0}^{\infty} p_n V^{*n}$, where $\{p_n : n \geq 0\}$ is a sequence of nonnegative numbers satisfying $\sum_{n=0}^{\infty} p_n = 1$. Suppose that there exists some $\varepsilon_0 > 0$ such that*

$$\sum_{n=0}^{\infty} p_n (C^*(V) - M_\gamma(V) + \varepsilon_0)^n < \infty, \quad (3.34)$$

then $U \in \mathcal{L}(\gamma) \cap \mathcal{OS}$.

3.3 Proofs of Theorem 3.1 and Proposition 3.4

In this section, we prove Theorem 3.1 and Proposition 3.4, respectively.

Proof of Theorem 3.1. We first prove (3.3). By $F \in \mathcal{L}$, Corollary 3.1 of Wang and Wang (2006) and Lemma 3.4, we know that

$$\begin{aligned} G(x + \Delta_T) &\sim (1-p)p^{-1}\mu^{-1}F_1(x + \Delta_T) \\ &\sim (1-p)p^{-1}\mu^{-1}T\overline{F}(x), \end{aligned} \quad (3.35)$$

hence $G \in \mathcal{L}_{\Delta_T}$. Thus by Corollary 1 of Asmussen et al. (2003) and Theorem 3.1 of Yu et al. (2010), we know that

$$\liminf W(x + \Delta_T)(G(x + \Delta_T))^{-1} = p(1-p)^{-1}. \quad (3.36)$$

So, (3.3) follows from (3.35) and (3.36).

Next, we prove (3.5). We know from (3.35), $F \in \mathcal{L} \cap \mathcal{OS}^*$, Lemmas 3.4 and 3.5 that $F_1 \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ and $G \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for all $0 < T < \infty$. By condition (3.4) and Lemma 3.5, we have

$$p(C^T(G) - 1) = p((1-p)p^{-1}\mu^{-1}(C^\otimes(F) - 2EX_1^+) + 1) < 1. \quad (3.37)$$

It follows from (3.1), (3.37), Lemmas 3.3 and 3.2 and the dominated convergence theorem that

$$\begin{aligned} \limsup \frac{W(x + \Delta_T)}{G(x + \Delta_T)} &\leq (1-p) \sum_{n=1}^{\infty} p^n \limsup \frac{G^{*n}(x + \Delta_T)}{G(x + \Delta_T)} \\ &\leq (1-p) \sum_{n=1}^{\infty} p^n \sum_{k=0}^{n-1} (C^T(G) - 1)^{n-1-k} \\ &= (1-p) \sum_{n=1}^{\infty} p^n ((C^T(G) - 1)^n - 1)(C^T(G) - 2)^{-1}, \end{aligned} \quad (3.38)$$

where, if $C^T(G) = 2$, then we define $((C^T(G) - 1)^n - 1)(C^T(G) - 2)^{-1} = n$ by continuity. By (3.37) and (3.38), we obtain that

$$\begin{aligned} \limsup \frac{W(x + \Delta_T)}{G(x + \Delta_T)} &\leq (1 - p)(C^T(G) - 2)^{-1} \left(\frac{p(C^T(G) - 1)}{1 - p(C^T(G) - 1)} - \frac{p}{1 - p} \right) \\ &= p(1 - p(C^T(G) - 1))^{-1}, \end{aligned}$$

thus (3.5) follows from (3.35), (3.37) and (3.28).

Now, we show that $W \in \mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}$ for any $0 < T < \infty$. By (3.3), (3.4), (3.5) and Lemma 3.5, we immediately get $W \in \mathcal{OS}_{\Delta_T}$. Next, we prove $W \in \mathcal{L}_{\Delta_T}$ for any $0 < T < \infty$. To this end, we denote the $-\gamma$ -transform of W and G by $U = W_{-\gamma}$ and $V = G_{-\gamma}$, respectively. From (3.1), we know that

$$M_{-\gamma}(W) = (1 - p)(1 - p_1)^{-1} > 1, \quad (3.39)$$

thus by (3.39), we have

$$M_\gamma(V) = (M_{-\gamma}(G))^{-1} < 1 \quad (3.40)$$

and

$$\overline{U}(x) = (1 - p_1) \sum_{n=1}^{\infty} p_1^n \overline{V^{*n}}(x), \quad (3.41)$$

where $0 < p_1 = pM_{-\gamma}(G) < 1$ for some γ large enough. By condition (3.35), Lemma 3.5 (ii) and (3.4), we have

$$C^T(G) - 2 = (1 - p)p^{-1}\mu^{-1}(C^\otimes(F) - 2EX_1^+) < (1 - p)p^{-1}. \quad (3.42)$$

From (3.42) and Lemma 3.6, we know that

$$p_1(C^*(V) - M_\gamma(V)) < 1.$$

Thus, by (3.41) and Lemma 3.7, $W_{-\gamma} \in \mathcal{L}(\gamma)$. According to Proposition 2.1 of Wang and Wang (2011), we know that $W \in \mathcal{L}_{\Delta_T}$ for any $0 < T < \infty$.

Finally, if $F \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}^*$, then by Corollary 3.2 of Wang et al. (2007), we have $W \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}$ for any $0 < T < \infty$. \square

Proof of Proposition 3.4. Let Y be a random variable with distribution $G \in \mathcal{L} \cap \mathcal{OS}^*$ supported on $(-\infty, \infty)$ and finite mean $EY = -\lambda < 0$. If $C^\otimes(G) < \lambda + 2EY^+$, we take distribution $F = G$ and $\mu = \lambda$, then distribution F satisfy the condition (3.4). Otherwise, if $C^\otimes(G) \geq \lambda + 2EY^+$, we set random variable $X = Y - a$ with distribution F and mean $\mu = -EX$ for some a large enough such that $C^\otimes(G) - 2EY^+ < \lambda + a$. It is easy to find that for any $h \in \mathcal{H}(F) = \mathcal{H}(G)$,

$$\begin{aligned} C^\otimes(F) - 2EX^+ &= \limsup \int_{h(x)}^{x-h(x)} \overline{F}(x-y) \overline{F}(y) dy (\overline{F}(x))^{-1} \\ &= \limsup \int_{h(x)}^{x-h(x)} \overline{G}(x-y) \overline{G}(y) dy (\overline{G}(x))^{-1} \\ &= C^\otimes(G) - 2EY^+ < \lambda + a = \mu, \end{aligned}$$

that is condition (3.4) holds. \square

4 Some applications

In this section, we give some practical applications of the results, concepts and methods of this paper to the renewal risk model and $M/G/1$ queue, respectively.

4.1 On the ruin distribution

In the renewal risk model, the claim sizes $Y_i, i \geq 1$ are independent, identically distributed r.v.s with a common non-degenerate distribution F_1 supported on $(0, \infty)$ and finite mean EY_1 ; the claims occur at the random instants of time $0 < T_1 < T_2 < \dots$ a.s., and the inter-arrival times $Z_i, i \geq 1$ are also independent, identically distributed r.v.s with a common non-degenerate distribution F_2 supported on $[0, \infty)$ and finite mean EZ_1 ; the sequences $\{Y_i : i \geq 1\}$ and $\{Z_i : i \geq 1\}$ are independent of each other; the initial capital and the premium income rate are denoted by $x \geq 0$ and $c > 0$, respectively; finally, the net profit condition $cEZ_1 > EY_1$ is required.

We denote r.v.s $X_i = Y_i - cZ_i, i \geq 1$, which have a common distribution F supported on $(-\infty, \infty)$ and mean EX_1 . Then the distribution W of the supremum M of a random walk $\{S_n = \sum_{i=0}^n X_i : n \geq 0\}$ is called the ruin distribution of the renewal risk model, and its tail distribution

$$\overline{W} = (1 - p) \sum_{n=1}^{\infty} p^n \overline{G^{*n}}$$

is called the ruin probability, where $0 < p = P(\tau_+ < \infty) < 1$ and $G = pF_+$. In particular, if Z_1 is exponentially distributed with intensity $(EZ_1)^{-1}$, then $p = EY_1(cEZ_1)^{-1}$ and $G = F_1^I$. See, for instance, Veravebeke (1977) or Embrechts et al. (1997).

Clearly, (3.1) holds for any $0 < T \leq \infty$ in this model. Further, we assume that $F \in \mathcal{L} \cap \mathcal{OS}^*$ and condition (3.4) is satisfied, then by Theorem 3.1, we can get the local asymptotic estimation for the ruin distribution W . In other words, if the local ruin probability is defined by $W(x + \Delta)$ for any $0 < T < \infty$, then its asymptotic estimation is obtained.

4.2 On the stationary distribution of the virtual waiting-time

For the sake of simplicity, we omit the detailed description of the $M/G/1$ queue. As Klüppelberg (1989) pointed out, the arrival rate and the service-time distribution were denoted by η and F_1 respectively, where F_1 has a finite mean $\mu(F_1)$. If $p = \eta\mu(F_1) < 1$, then the stationary distribution W of the virtual waiting-time can be written as

$$W = (1 - p) \sum_{n=0}^{\infty} p^n G^{*n},$$

where $G = F_1^I$. Thus (3.1) holds for any $0 < T \leq \infty$.

Further, we assume that $F_1 \in \mathcal{L} \cap \mathcal{OS}^*$ and $C^{\otimes}(F_1) < (1 + p^{-1})\mu(F_1)$, then for any $0 < T < \infty$, $G(x + \Delta) \sim T(\mu(F_1))^{-1} \overline{F_1}(x)$. Therefore, by the proof of Theorem 3.1 and Lemma 3.5 (ii), we can also get the local asymptotic estimation for the stationary

distribution W of the virtual waiting-time as follows:

$$\begin{aligned} Tp(1-p)^{-1}(\mu(F_1))^{-1} &= \liminf W(x+\Delta)(\overline{F_1}(x))^{-1} \\ &\leq \limsup W(x+\Delta)(\overline{F_1}(x))^{-1} \\ &\leq Tp\left(1-p(\mu(F_1))^{-1}(C^\otimes(F_1)-\mu(F_1))\right)^{-1}(\mu(F_1))^{-1}. \end{aligned}$$

Particularly, if $F_1 \in \mathcal{S}^*$, that is $C^\otimes(F_1) = 2\mu(F_1)$, then we have

$$W(x+\Delta)(\overline{F_1}(x))^{-1} \sim Tp(1-p)^{-1}(\mu(F_1))^{-1}.$$

In addition to the above two examples, in a number of areas of applied probability, such as branching processes, infinitely divisible distribution and so on, the distribution of the research objects can be written in the form of (3.1), or the form of other compound distributions, so the results of this paper can also be applied to these areas. Here, we omit the details.

5 Proofs of Examples

5.1 Proof of $G_m \in \mathcal{S} \cap \mathcal{OS}^* \setminus \mathcal{S}^*$ in Example 2.1.

It is not hard to see that for all $m \geq 1$ and $x \geq x_1$, one has

$$x^{-2\alpha+m^{-1}} \leq \overline{F}(x) \leq 2^\alpha x^{-\alpha}. \quad (5.1)$$

Since $\alpha \in (m^{-1}, 1+m^{-1})$, by $\alpha > 1$ and (5.1), we know that the distribution G_m has a finite mean $m(G_m) = \mu$. Denote

$$f(x) = \sum_{n=1}^{\infty} (x_n^{-\alpha-1} - x_n^{-2\alpha-1+m^{-1}}) \mathbf{1}(x_n < x < 2x_n).$$

We first prove that $G_m \notin \mathcal{S}^*$. To this end, for all $x \geq 0$, denote

$$H(x) = (\overline{G_m}(x))^{-1} \int_0^x \overline{G_m}(x-y) \overline{G_m}(y) dy \quad (5.2)$$

By (2.1), one has

$$\begin{aligned} H(2x_n) &= 2(\overline{F}^m(2x_n))^{-1} \int_{x_n}^{2x_n} \overline{F}^m(2x_n-y) \overline{F}^m(y) dy \\ &= 2 \int_0^{x_n} \overline{F}^m(y) (1 + (x_n^{\alpha-m^{-1}-1} - x_n^{-1})y)^m dy \\ &= 2 \int_0^{x_n} \overline{F}^m(y) dy + 2 \int_0^{x_n} \overline{F}^m(y) \left((1 + (x_n^{\alpha-m^{-1}-1} - x_n^{-1})y)^m - 1 \right) dy. \end{aligned} \quad (5.3)$$

By (5.1) and (2.1),

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n^{\alpha m - m - 1} \int_0^{x_n} \overline{F}^m(y) y^m dy &= \lim_{n \rightarrow \infty} x_n^{\alpha m - m - 1} \int_{2x_n - 1}^{x_n} \overline{F}^m(y) y^m dy \\ &= (m+1)^{-1}. \end{aligned} \quad (5.4)$$

And for all $t = 1, \dots, m-1$,

$$\lim_{n \rightarrow \infty} x_n^{(\alpha - m^{-1} - 1)t} \int_0^{x_n} \overline{F}^m(y) y^t dy = 0. \quad (5.5)$$

By (5.3)-(5.5),

$$\lim_{n \rightarrow \infty} H(2x_n) = 2\mu + 2(m+1)^{-1}, \quad (5.6)$$

thus $G_m \notin \mathcal{S}^*$.

Next, we prove $G_m \in \mathcal{OS}^*$. We estimate $H(x)$ in the cases $x_n \leq x < \frac{3}{2}x_n$, $\frac{3}{2}x_n \leq x < 2x_n$ and $2x_n \leq x < x_{n+1}$, $n \geq 1$, respectively. When $x \in [x_n, 3 \cdot 2^{-1}x_n)$, by (5.1) and (2.1),

$$\begin{aligned} H(x) &\leq 2^{m\alpha+1} (\overline{F}^m(3 \cdot 2^{-1}x_n))^{-1} \int_{\frac{x}{2}}^x \overline{F}^m(x-y) y^{-m\alpha} dy \\ &\leq 2^{2m\alpha+1+m} \int_0^{3 \cdot 4^{-1}x_n} \overline{F}^m(y) dy \leq 2^{2m\alpha+1+m} \mu. \end{aligned} \quad (5.7)$$

When $x \in [3 \cdot 2^{-1}x_n, 2x_n)$, by (2.1), (5.3) and (5.6),

$$\begin{aligned} H(x) &= 2(\overline{F}^m(x))^{-1} \left(\int_{\frac{x}{2}}^{x_n} + \int_{x_n}^x \right) \overline{F}^m(x-y) \overline{F}^m(y) dy \\ &= 2(\overline{F}^m(x))^{-1} \left(x_n^{-m\alpha} \int_{x-x_n}^{2^{-1}x} \overline{F}^m(x-y) \overline{F}^m(y) dy + \int_0^{x-x_n} \overline{F}^m(x-y) \overline{F}^m(y) dy \right) \\ &\leq 2(\overline{F}^m(2x_n))^{-1} x_n^{-m\alpha} \int_{2^{-1}x_n}^{x_n} \overline{F}^m(y) dy \\ &\quad + 2 \int_0^{x-x_n} \overline{F}^m(y) \left(1 + (x_n(1 - x_n^{-\alpha+m-1}))^{-1} - (x-x_n)^{-1} y \right)^m dy \\ &\leq 1 + 2 \int_0^{x_n} \overline{F}^m(y) \left(1 + (x_n^{\alpha-m-1-1} - x_n^{-1}) y \right)^m dy \\ &= 1 + H(2x_n) \rightarrow 1 + 2\mu + 2(m+1)^{-1}, \quad n \rightarrow \infty. \end{aligned} \quad (5.8)$$

When $x \in [2x_n, x_{n+1})$, by (2.1) and (5.3),

$$\begin{aligned} H(x) &\leq 2(\overline{F}^m(x))^{-1} \left(\int_{x_n}^{2x_n} + \int_{2x_n}^x \right) \overline{F}^m(x-y) \overline{F}^m(y) dy \\ &\leq H(2x_n) + 2 \int_0^{x-2x_n} \overline{F}^m(y) dy \rightarrow 4\mu + 2(m+1)^{-1}, \quad n \rightarrow \infty. \end{aligned} \quad (5.9)$$

It follows from (5.7)-(5.9) that $G_m \in \mathcal{OS}^*$.

Finally, we prove $G_m \in \mathcal{S}$. By (5.1), we have

$$\overline{F}^{2m}(2^{-1}x)(\overline{F}^m(x))^{-1} \leq 2^{4m\alpha} x^{-1} \rightarrow 0. \quad (5.10)$$

By (5.10) and

$$\overline{G_m^{*2}}(x) = 2\overline{G_m}(x) - \overline{G_m^2}(2^{-1}x) + 2 \int_{2^{-1}x}^x \overline{G_m}(x-y) dG_m(y),$$

we know that in order to prove $G_m \in \mathcal{S}$, it suffices to prove

$$\begin{aligned} T(x) &= 2(\overline{F^m}(x))^{-1} \int_{2^{-1}x}^x \overline{F^m}(x-y) d(1 - \overline{F^m}(x)) \\ &= 2m(\overline{F^m}(x))^{-1} \int_{2^{-1}x}^x \overline{F^m}(x-y) \overline{F^{m-1}}(y) f(y) dy \rightarrow 0. \end{aligned} \quad (5.11)$$

Clearly, $T(x_n) = 0, n \geq 1$. By (2.1) and (5.5), we have

$$\begin{aligned} T(2x_n) &= 2m\overline{F^m}(2x_n) \int_0^{x_n} \overline{F^m}(y) \overline{F^{m-1}}(2x_n - y) f(2x_n - y) dy \\ &= 2m(x_n^{\alpha-m^{-1}-1} - x_n^{-1}) \int_0^{x_n} \overline{F^m}(y) (1 + (x_n^{\alpha-m^{-1}-1} - x_n^{-1})y)^{m-1} dy \\ &\leq 2mx_n^{\alpha-m^{-1}-1} \int_0^{x_n} \overline{F^m}(y) (1 + x_n^{\alpha-m^{-1}-1}y)^{m-1} dy \\ &= 2mx_n^{\alpha-m^{-1}-1} \left(\int_0^{x_n} \overline{F^m}(y) dy + \int_0^{x_n} \overline{F^m}(y) ((1 + x_n^{\alpha-m^{-1}-1}y)^{m-1} - 1) dy \right) \\ &\rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (5.12)$$

In the following, we prove (5.11) in the cases $x_n \leq x < 2x_n$ and $2x_n \leq x < x_{n+1}$, $n \geq 1$, respectively. When $x \in [x_n, 2x_n)$, by (2.1) and (5.12),

$$\begin{aligned} T(x) &= 2m(\overline{F^m}(x))^{-1} \int_{x_n}^x \overline{F^m}(x-y) \overline{F^{m-1}}(y) f(y) dy \\ &= 2m(\overline{F^m}(x))^{-1} \int_0^{x-x_n} \overline{F^m}(y) \overline{F^{m-1}}(x-y) f(x-y) dy \\ &\leq 2m(x_n^{\alpha-m^{-1}-1} - x_n^{-1}) \int_0^{x-x_n} \overline{F^m}(y) \left(1 + (x_n(1 - x_n^{-\alpha+m^{-1}})^{-1} - (x-x_n))^{-1}y \right)^{m-1} dy \\ &\leq T(2x_n) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (5.13)$$

When $x \in [2x_n, x_{n+1})$, by (2.1) and (5.12),

$$\begin{aligned} T(x) &\leq 2m(\overline{F^m}(2x_n))^{-1} \int_{x_n}^{2x_n} \overline{F}(2x_n - y) \overline{F^{m-1}}(y) f(y) dy \\ &= T(2x_n) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned} \quad (5.14)$$

According to (5.13) and (5.14), (5.11) holds, thus $G_m \in \mathcal{S}$.

In summary, we have $G_m \in (\mathcal{S} \cap \mathcal{OS}^*) \setminus \mathcal{S}^*$. □

5.2 Proof of $G_m \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}$ in Example 2.2.

Still let f be the density of F , when $x \geq x_1$, it is easily seen that

$$x^{-2^{-1}(\alpha+1)} \leq \overline{F}(x) \leq 2^\alpha x^{-2^{-1}\alpha} \text{ and } f(x) \leq 2^{\alpha+1} x^{-2^{-1}\alpha-1} \quad (5.15)$$

Moreover, one can easily find that the distribution G_m has a finite mean for $m \geq 1$, and we still denote it by μ .

Firstly, by (5.15) we have,

$$(\overline{F}(x))^{-1} f(x) \leq 2^{\alpha+1} x^{-2^{-1}} \rightarrow 0, \quad ,$$

thus $F \in \mathcal{L}$, so $G_m \in \mathcal{L}$. Next, we prove that $G_m \in \mathcal{OS}^*$. Let $H(x), x \geq 0$ be the same as in (5.2), it is easily seen that

$$\begin{aligned} H(4x_n^2) &= 2(\overline{F}^m(4x_n^2))^{-1} \int_0^{2x_n^2} \overline{F}^m(y) \overline{F}^m(4x_n^2 - y) dy \\ &= 2 \int_0^{2x_n^2} \overline{F}^m(y) (1 + (1 - x_n^{-1})(4x_n^2 - y)^{2^{-1}} + (2x_n)^{-1}y)^m dy \\ &\leq 2 \int_0^{2x_n^2} \overline{F}^m(y) (1 + (2x_n)^{-1}y)^m dy \\ &\leq 2x_1^2 (1 + (2x_n)^{-1}x_1^2)^m + 2^{m\alpha+1} \int_{x_1^2}^{\infty} y^{m-\frac{m\alpha}{2}} (y^{-1} + (2x_n)^{-1})^m dy \\ &\leq 2x_1^2 (1 + x_1^2)^m + 2^{m\alpha+1} (2^{-1}m\alpha - m - 1)^{-1} x_1^{2m-m\alpha} (1 + x_1^2)^m < \infty. \end{aligned}$$

Just as in Example 2.1, we deal with $H(x)$ in three cases $x_n^2 \leq x < 2x_n^2$, $2x_n^2 \leq x < 4x_n^2$ and $4x_n^2 \leq x < x_{n+1}^2$, $n \geq 1$, respectively. When $x \in [x_n^2, 2x_n^2)$, just as (5.7), by (2.2) and variable substitution, we have

$$\begin{aligned} H(x) &= 2(\overline{F}^m(x))^{-1} \int_{\frac{x}{2}}^x \overline{F}^m(x-y) \overline{F}^m(y) dy \\ &\leq 2(\overline{F}^m(2x_n^2))^{-1} \int_{\frac{x}{2}}^x \overline{F}^m(x-y) x_n^{-m\alpha} dy \\ &\leq 2(2 - 2^{2^{-1}})^{-m} \int_0^{x_n^2} \overline{F}^m(y) dy < \infty. \end{aligned} \quad (5.16)$$

When $x \in [2x_n^2, 4x_n^2)$, just as (5.8), by (2.2), we have

$$\begin{aligned}
H(x) &= 2(\overline{F}^m(x))^{-1} \int_0^{2^{-1}x} \overline{F}^m(y) \overline{F}^m(x-y) dy \\
&= 2(\overline{F}^m(x))^{-1} \int_0^{2^{-1}x} \overline{F}^m(y) \left(\overline{F}(x) + (x_n^{-\alpha-1} - x_n^{-\alpha-2})(x^{2^{-1}} + (x-y)^{2^{-1}})^{-1}y \right)^m dy \\
&= 2 \int_0^{2^{-1}x} \overline{F}^m(y) \left(1 + (x_n^{-\alpha-1} - x_n^{-\alpha-2})(\overline{F}(x)(\sqrt{x} + \sqrt{x-y}))^{-1}y \right)^m dy \\
&\leq 2 \int_0^{2x_n^2} \overline{F}^m(y) (1 + (2^{2^{-1}}x_n)^{-1}y)^m dy \\
&\leq 2x_1^2(1 + (2^{2^{-1}}x_n)^{-1}x_1^2)^m + 2^{m\alpha+1} \int_{x_1^2}^{\infty} y^{m-\frac{m\alpha}{2}} (y^{-1} + (2^{2^{-1}}x_n)^{-1})^m dy \\
&\leq 2x_1^2(1 + x_1^2)^m + 2^{m\alpha+1}(2^{-1}m\alpha - m - 1)^{-1}x_1^{2m-m\alpha}(1 + x_1^2)^m < \infty.
\end{aligned} \tag{5.17}$$

When $x \in [4x_n^2, x_{n+1}^2)$, just as (5.9), by (2.2), we have

$$\begin{aligned}
H(x) &\leq 2(\overline{F}^m(x))^{-1} \left(\int_{2x_n^2}^{4x_n^2} + \int_{4x_n^2}^x \right) \overline{F}^m(x-y) \overline{F}^m(y) dy \\
&\leq H(4x_n^2) + 2 \int_0^{x-4x_n^2} \overline{F}^m(y) dy < \infty.
\end{aligned} \tag{5.18}$$

According to (5.16) and (5.17), (5.18) holds, that is $G_m \in \mathcal{OS}^*$.

Finally, we prove $G_m \notin \mathcal{S}$. Since

$$\overline{G}_m^2(2x_n^2)(\overline{G}_m(4x_n^2))^{-1} = x_n^{m-m\alpha}((2 - 2^{2^{-1}}) + (2^{2^{-1}} - 1)x_n^{-1})^{2m} \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore, in order to prove $G_m \notin \mathcal{S}$, we only need to prove

$$\liminf_{x \rightarrow \infty} T(x) = \liminf_{x \rightarrow \infty} (\overline{G}_m(x))^{-1} \int_{2^{-1}x}^x \overline{G}_m(x-y) G_m(dy) > 0. \tag{5.19}$$

In fact,

$$\begin{aligned}
T(4x_n^2) &= m(\overline{F}^m(4x_n^2))^{-1} \int_{2x_n^2}^{4x_n^2} \overline{F}^m(4x_n^2 - y) \overline{F}^{m-1}(y) f(y) dy \\
&= 2^{-1}m(1 - x_n^{-1}) \int_0^{2x_n^2} \overline{F}^m(y) \left(1 + (1 - x_n^{-1})(\sqrt{4x_n^2 - y} + 2x_n)^{-1}y \right)^{m-1} y^{-2^{-1}} dy \\
&\geq 2^{-1}m(1 - x_n^{-1}) \int_{x_1^2}^{2x_n^2} \overline{F}^m(y) y^{-\frac{1}{2}} dy \\
&\geq m(m\alpha + m - 1)^{-1}(1 - x_n^{-1})(x_1^{1-m\alpha-m} - (2x_n^2)^{-2^{-1}(m\alpha+m-1)}) \\
&\rightarrow m(m\alpha + m - 1)^{-1}x_1^{1-m\alpha-m} > 0, \quad n \rightarrow \infty,
\end{aligned}$$

that is, (5.19) holds, that is $G_m \notin \mathcal{S}$.

In summary, we have $G_m \in (\mathcal{L} \cap \mathcal{OS}^*) \setminus \mathcal{S}$. □

5.3 Proof of $G_m^I \in (\mathcal{L}_{\Delta_T} \cap \mathcal{OS}_{\Delta_T}) \setminus \mathcal{S}_{\Delta_T}$ in Example 2.3.

This conclusion follows directly from $G_m \in (\mathcal{S} \cap \mathcal{OS}^*) \setminus \mathcal{S}^*$, Lemmas 3.4, 3.5 and Lemma 4.2 of Wang et al. (2007). \square

5.4 Proof of $G_m^I \in \mathcal{OS}_{\Delta_T} \setminus \mathcal{S}_{\Delta_T}$ for all $0 < T < \infty$ in Example 2.4.

By Proposition 2.1 of Chen et al. (2013), we have $F_\gamma \in \mathcal{L}_{\Delta_T} \setminus \mathcal{S}_{\Delta_T}$ for all $T > 0$. We now prove $F_\gamma \in \mathcal{OS}_{\Delta_T}$. Since $F_i \in \mathcal{S}(\gamma) \subset \mathcal{OS}$, $i = 1, 2$, so $F = F_1 * F_2 \in \mathcal{OS}$ by Proposition 6.1 of Yu and Wang (2013). By $F \in \mathcal{OS}$ and (2.1) of Chen et al. (2013), we have

$$\limsup F_\gamma^{*2}(x + \Delta_T)(F_\gamma(x + \Delta_T))^{-1} = (M_\gamma(F))^{-1} \limsup \overline{F^{*2}}(x)(\overline{F}(x))^{-1} < \infty,$$

thus $F_\gamma \in \mathcal{OS}_{\Delta_T}$. \square

5.5 Proofs of $G_m \notin \mathcal{OS}^*$, $G_m^I \in (\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$ and $G_m^I \in \mathcal{L}_{\Delta_T} \setminus \mathcal{OS}_{\Delta_T}$ for all $0 < T < \infty$ in Example 2.5.

To show that $G_m \notin \mathcal{OS}^*$, we denote the density of F by f . Consider the following quantity

$$I(x) = (\overline{G_m}(2x_n))^{-1} \int_{x_n}^{2x_n - 2x_{n-1}} \overline{G_m}(2x_n - y) \overline{G_m}(y) dy.$$

Since $x_n \leq y \leq 2x_n - 2x_{n-1} \leq 2x_n$, we have $2x_{n-1} \leq 2x_n - y \leq x_n$, $n \geq 1$, so by (2.3),

$$\begin{aligned} I(x) &= 2^{2m\alpha} x_n^{m\alpha} \int_{x_n}^{2x_n - 2x_{n-1}} (x_n^{-\alpha} - f(x_n)(y - x_n))^m dy \\ &= 2^{2m\alpha} (m+1)^{-1} x_n^{m\alpha} (x_n - 2x_{n-1}) \sum_{i=0}^m x_n^{-\alpha i} (x_n^{-\alpha} - f(x_n)(x_n - 2x_{n-1}))^{m-i} \\ &\geq 2^{2m\alpha} (m+1)^{-1} (x_n - 2x_{n-1}) \rightarrow \infty, \quad n \rightarrow \infty, \end{aligned}$$

thus $G_m \notin \mathcal{OS}^*$.

Since $G_m \in \mathcal{L} \setminus \mathcal{OS}^*$, we immediately get $G_m^I \in \mathcal{L}_{\Delta_T} \setminus \mathcal{OS}_{\Delta_T}$ by Lemmas 3.4 and 3.5.

Now we show that $G_m^I \notin \mathcal{S}$. By (2.3), it is obvious that

$$x^{-2\alpha} \leq \overline{F}(x) \leq x^{-\alpha}. \quad (5.20)$$

By (5.20), we have

$$\begin{aligned} (\overline{G_m^I}(x_n))^2 (\overline{G_m^I}(2x_n))^{-1} &\geq \mu^{-1} \left(\int_{x_n}^{2x_n} \overline{F}^m(y) dy \right)^2 \left(\int_{2x_n}^{\infty} \overline{F}^m(y) dy \right)^{-1} \\ &\geq \mu^{-1} (m+1)^{-2} 2^{2m\alpha-2} (1 - m^{-1} \alpha^{-1}) > 0, \end{aligned}$$

thus by (2.4) of Murphree (1989), we complete the proof.

To prove $G_m^I \in \mathcal{OS}$, we consider

$$\begin{aligned}
& \int_{\frac{x}{2}}^x \overline{G}_m^I(x-y) dG_m^I(y) = \mu^{-1} \int_{\frac{x}{2}}^x \overline{F}^m(y) \int_{x-y}^{\infty} \overline{F}^m(z) dz dy \\
&= \mu^{-1} \left(\int_{\frac{x}{2}}^x \overline{F}^m(y) \int_{x-y}^{\frac{x}{2}} \overline{F}^m(z) dz dy + \left(\int_{\frac{x}{2}}^x \overline{F}^m(y) dy \right)^2 + \int_{\frac{x}{2}}^x \overline{F}^m(y) \int_x^{\infty} \overline{F}^m(z) dz dy \right) \\
&= \mu^{-1} (I_1(x) + I_2(x) + I_3(x)). \tag{5.21}
\end{aligned}$$

We first estimate $I_1(x)$ in the cases $x_n \leq x < 4x_n$ and $4x_n \leq x < x_{n+1}$, $n \geq 1$, respectively. When $x \in [x_n, 4x_n)$, by (2.3) and (5.20), we have

$$\begin{aligned}
& \left(\int_x^{\infty} \overline{F}^m(y) dy \right)^{-1} I_1(x) \leq \left(\int_{4x_n}^{x_{n+1}} \overline{F}^m(y) dy \right)^{-1} \int_{2^{-1}x}^x x_n^{-m\alpha} \int_{x-y}^{2^{-1}x} \overline{F}^m(z) dz dy \\
&\leq 2^{2m\alpha-2} (x_n^{2-m\alpha} - x_n^{1-m\alpha})^{-1} \int_0^{2^{-1}x} \int_y^{2^{-1}x} \overline{F}^m(z) dz dy \\
&\leq 2^{2m\alpha-1} (x_n^{2-m\alpha})^{-1} \left(\int_{x_1}^{2x_n} \int_y^{2x_n} z^{-m\alpha} dz dy + \int_0^{x_1} \int_{x_1}^{2x_n} z^{-m\alpha} dz dy + \int_0^{x_1} \int_y^{x_1} dz dy \right) \\
&\leq 2^{m\alpha+1} (m\alpha-1)^{-1} (2-m\alpha)^{-1} + ((m\alpha-1)^{-1} x_1^{2-m\alpha} + x_1^2) < \infty. \tag{5.22}
\end{aligned}$$

Similarly, when $x \in [4x_n, x_{n+1})$, we have

$$\begin{aligned}
& \left(\int_x^{\infty} \overline{F}^m(y) dy \right)^{-1} I_1(x) \leq \left(\int_{x_{n+1}}^{2x_{n+1}} \overline{F}^m(y) dy \right)^{-1} \int_{2^{-1}x}^x (2x_n)^{-2m\alpha} \int_{x-y}^{2^{-1}x} \overline{F}^m(z) dz dy \\
&\leq \frac{m+1}{2x_n} \left(\int_{x_1}^{\frac{x_{n+1}}{2}} \int_y^{\frac{x_{n+1}}{2}} z^{-m\alpha} dz dy + \int_0^{x_1} \int_{x_1}^{\frac{x_{n+1}}{2}} z^{-m\alpha} dz du + \int_0^{x_1} \int_y^{x_1} dz dy \right) \\
&\leq (m+1) ((m\alpha-1)^{-1} (2-m\alpha)^{-1} 2^{-m\alpha} + ((m\alpha-1)^{-1} x_1^{2-m\alpha} + x_1^2)) < \infty. \tag{5.23}
\end{aligned}$$

Next, we estimate $I_2(x)$. When $x \in [x_n, x_{n+1})$, by (5.20), we have

$$\begin{aligned}
& \left(\int_x^{\infty} \overline{F}^m(y) dy \right)^{-1} \left(\int_{2^{-1}x}^x \overline{F}^m(y) dy \right)^2 \leq \left(\int_{x_{n+1}}^{2x_{n+1}} \overline{F}^m(y) dy \right)^{-1} \left(\int_{2^{-1}x_n}^{x_{n+1}} \overline{F}^m(y) dy \right)^2 \\
&\leq (m+1) \left(2^{-4+2m\alpha} 9 + 3x_n^{1-m\alpha} + (2x_n)^{2-2m\alpha} \right) < \infty. \tag{5.24}
\end{aligned}$$

Finally, for $I_3(x)$, it is obvious that

$$\left(\int_x^{\infty} \overline{F}^m(y) dy \right)^{-1} I_3(x) \leq \int_{2^{-1}x}^x \overline{F}^m(y) dy \rightarrow 0. \tag{5.25}$$

By (5.21)-(5.25), we get $G_m^I \in \mathcal{OS}$. \square

Acknowledgement. The authors are grateful to the two referees for his/her careful reading and valuable comments and suggestions, which greatly improve the original version of this paper.

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